Renormalization and chaos in the logistic map
Logistic map

Many features of non-Hamiltonian chaos can be seen in this simple map (and other similar one dimensional maps).

Why? Universality.

Period doubling
Intermittency
Crisis
Ergodicity
Strange attractors
Periodic/Aperiodic mix
Topological Cantor set
Time series

Superstable points; convergence to attractor very rapid
Critical slowing down near a bifurcation; convergence very slow
Superstable points; convergence to attractor very rapid
Critical slowing down near a bifurcation; convergence very slow
Bifurcation to a two cycle attractor
A series of period doublings of the attractor occur for increasing values of $r$, the “biotic potential” as it is sometimes known for the logistic map.
At a critical value of $r$ the dynamics become *aperiodic*. However, note that initially close trajectories remain close. Dynamics are not ergodic.
At $r = 4$ one finds aperiodic motion, in which initially close trajectories exponentially diverge.
Self similarity, intermittency, “crisis”
Other one dimensional maps

Other one dimensional maps show a rather similar structure; LHS is the sine map and RHS one that I made up randomly.

In fact the structure of the periodic cycles with $r$ is universal.
Universality

$2^{nd}$ order phase transitions: details, i.e. interaction form, *does not matter near the phase transition.*

Depend only on (e.g.) symmetry of order parameter, dimensionality, range of interaction

Can be calculate from simple models that share these feature with (complicated) reality.
The period doubling regime

- Derivative identical for all points in cycle.
- All points become unstable simultaneously; period doubling topology can only be that shown above.

\[ f_r^{2n} \equiv f_r(\ldots f_r(x_0)) \]

\[ \frac{df_r^n}{dx} = \prod_{i=0}^{n-1} f'_r(x_i) \]

\[ f_r^2(x_0) = f'_r(f_r(x_0)) f'_r(x_0) = f'_r(x_1) f'_r(x_0) \]
The period doubling regime

Superstable cycles so called due to accelerated form of convergence; in Taylor expansion first order term vanishes.

\[ d_n = f_{R_n}^{2^{n-1}}(1/2) - 1/2 \]

Use continuity

\[ \delta x_n = \lambda^n \delta x_0 \]

\[ \delta x_n = \gamma^{2^{n-1}} (\delta x_0)^{2^n} \]
The period doubling regime

These constants have been measured in many experiments!
Renormalisation

$\quad d_n = f_{R_n}^{2^{n-1}}(0)$

Change coordinates such that superstable point is at origin.
Renormalisation

\[ \alpha^2 f_{R_2}^4 \left( \frac{x}{\alpha^2} \right) \approx \alpha f_{R_1}^2 \left( \frac{x}{\alpha} \right) \]
Renormalisation
Renormalisation
Universal mapping functions

\[
\lim_{n \to \infty} \alpha^n f_{R_n}^{2^n} \left( \frac{x}{\alpha^n} \right) = g_0(x)
\]

\[
\lim_{n \to \infty} \alpha^n f_{R_{n+i}}^{2^n} \left( \frac{x}{\alpha^n} \right) = g_i(x)
\]

Each of these functions is an i cycle attractor.

\[
\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \to \delta
\]

\[
\frac{d_n}{d_{n+1}} \to \alpha
\]

\[
r_\infty - r_n = \text{const.} \delta^{-n}
\]

\[
\lim_{n \to \infty} \alpha^n d_{n+1} = d_1
\]

\[
d_{n+1} = f_{R_{n+1}}^{2^n}(0)
\]

\[
\lim_{n \to \infty} \alpha^n f_{R_{n+1}}^{2^n}(0) = g_1(0)
\]
Doubling operator for g's

\[ g_{i-1} = \alpha g_i(g_i(x/\alpha)) \]
\[ = \mathbf{T}g_i \quad \text{Doubling operator in function space g} \]

\[ g_{i-1}(x) = \lim_{n \to \infty} \alpha^n f_{R_{n+i-1}}^{2^{n-1}} \left( f_{R_{n+i-1}}^{2^{n-1}} \left( \frac{x}{\alpha^n} \right) \right) \]
\[ = \lim_{n \to \infty} \alpha^{n-1} f_{R_{n-1+i}}^{2^{n-1}} \left( \frac{1}{\alpha^{n-1}} \alpha^{n-1} f_{R_{n-1+i}}^{2^{n-1}} \left( \frac{1}{\alpha} \frac{x}{\alpha^{n-1}} \right) \right) \]
\[ = \lim_{m \to \infty} \alpha^m f_{R_{m+i}}^{2^m} \left( \frac{1}{\alpha^m} \alpha^m f_{R_{m+i}}^{2^m} \left( \frac{1}{\alpha} \frac{x}{\alpha^m} \right) \right) \]

\[ \mathbf{T}g_\infty = g_\infty \quad g_\infty(x) = g(x) \]

\[ g(x) = \alpha g(g(x/\alpha)) \]

\[ g(0) = 1 \quad \alpha = -2.5029078750958928 \ldots \]
Eigenvalues of the doubling operator

\[ f_r = f_{\infty} + (r - R_{\infty})\delta f \]

\[ \delta f = \left. \frac{\partial f_r(x)}{\partial r} \right|_{R_{\infty}} \]

\[ T_{f_r} = T[f_{\infty} + (r - R_{\infty})\delta f] \]
\[ = \alpha f_{\infty} \left( f_{\infty} \left( \frac{x}{\alpha} \right) + (r - R_{\infty})\delta f \left( \frac{x}{\alpha} \right) \right) + \alpha(r - R_{\infty})\delta f \left( f_{\infty} \left( \frac{x}{\alpha} \right) + (r - R_{\infty})\delta f \left( \frac{x}{\alpha} \right) \right) \]
\[ = \alpha f_{\infty} \left( f_{\infty} \left( \frac{x}{\alpha} \right) \right) + \alpha(r - R_{\infty})f'_{\infty} \left( f_{\infty} \left( \frac{x}{\alpha} \right) \right) \delta f \left( \frac{x}{\alpha} \right) + \alpha(r - R_{\infty})\delta f \left( f_{\infty} \left( \frac{x}{\alpha} \right) \right) \]

\[ T_{f_r} = T_{f_{R_{\infty}}} + (r - R_{\infty})L_{f_{R_{\infty}}} \delta f \]

\[ L_{f}h = \alpha f' \left( f \left( \frac{x}{\alpha} \right) \right) h + \alpha h \left( f \left( \frac{x}{\alpha} \right) \right) \]

Linearize previous equation
Eigenvalues of the doubling operator

\[ T^n f_r = T^n f_{R \infty} + (r - R_\infty) L_{T^{n-1} f_{R \infty}} \cdots L_{f_{R \infty}} \delta f \]

\[ T^n f_{R \infty} = \alpha^n f_{R \infty}^2 \left( \frac{x}{\alpha^n} \right) \]
\[ = g(x) \]

\[ T^n f_r(x) = g(x) + (r - R_\infty) L_g^n \delta f(x) \quad n \gg 1 \]

\[ L_g \phi_\nu = \lambda \phi_\nu \]
\[ \delta f = \sum_\nu c_\nu \phi_\nu \]

\[ L_g^n \delta f = \sum_\nu c_\nu \lambda_\nu^n \phi_\nu \]

This was an assumption of Feigenbaum’s; proved later (1982)
Eigenvalues of the doubling operator

\[ r = R_n \]
\[ x = 0 \]

\[ T^n f_{R_n}(0) = \alpha^n f_{R_n}^{2^n}(0) = 0 \]

\[ T^n f_r(x) = g(x) + (r - R_\infty)c_1 \phi_1(x)\lambda_1^n \]

\[ (R_\infty - R_n) = \frac{1}{c_1 \phi_1(x)} \lambda_1^{-n} \quad \Rightarrow \quad \lambda_1 = \delta \]

\[ L_g \phi_1 = \alpha g'(g(x/\alpha))\phi_1 + \alpha \phi_1(g(x/\alpha)) = \delta \phi_1 \]

\[ \delta = 4.66920160910 \ldots \]
Liapunov exponents

$$\left| \delta_n \right| = \left| \delta_0 \right| e^{n \lambda} \quad \left| \delta_0 \right| \to 0$$

$$n \to \infty$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left| \lim_{\delta_0 \to 0} \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right|$$

$$= \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{df^n(x_0)}{dx_0} \right|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{df(x_i)}{dx_0} \right|$$

$$\lambda = \frac{1}{p} \sum_{i=0}^{p-1} \ln \left| \frac{df(x_i)}{dx_0} \right|$$

$$\lambda = 0$$

$$\lambda \to -\infty$$

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\[ |\delta_0| \to 0 \]
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\[ \lambda = \frac{1}{p} \sum_{i=0}^{p-1} \ln \left| \frac{df(x_i)}{dx_0} \right| \quad \lambda = 0 \]
\[ \lambda \to -\infty \]
Fully developed chaos at $r = 4$

$$x_{n+1} = 4x_n(1 - x_n)$$

$$x = \sin^2 \pi y = 1/2(1 - \cos 2\pi y)$$

$$y_{n+1} = 2y_n \mod 1$$

$$y_{n+1} = \begin{cases} 
2y_n, & \text{if } y_n < 1/2 \\
2y_n - 1, & \text{if } y_n \geq 1/2 
\end{cases}$$

$$x = \sum_{\nu=1}^{\infty} a_\nu 2^{-\nu}$$

Change variables, leads to exact solution for this case. (Other cases: $r=-2$, $r=+2$).

Bit shift map

Any number periodic in base 2 leads to periodic dynamics of the logistic map

Irrational numbers lead to non-periodic dynamics.
Ergodicity at $r = 4$

\[ \rho(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} \delta(x - f^i(x_0)) \quad \text{Invariant density} \]

\[ \rho(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} \quad \text{Ergodic: all points visited} \]

Strange attractor:

- Fractal
- Nearby points diverge under map flow
- All points visited: ergodic i.e., topologically mixing

Strange attractor of logistic map (and similar one dimensional maps) is a “topological Cantor set”